Computational Economics **18:** 287–316, 2001.

Digital Portfolio Theory

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Abstract. The Modern Portfolio Theory of Markowitz maximized portfolio expected return subject to holding total portfolio variance below a selected level. Digital Portfolio Theory is an extension of Modern Portfolio Theory, with the added dimension of memory. Digital Portfolio Theory decomposes the portfolio variance into independent components using the signal processing decomposition of variance. The risk or variance of each security's return process is represented by multiple periodic components. These periodic variance components are further decomposed into systematic and unsystematic parts relative to a reference index. The Digital Portfolio Theory model maximizes portfolio expected return subject to a set of linear constraints that control systematic, unsystematic, calendar and non-calendar variance. The paper formulates a single period, digital signal processing, portfolio selection model using cross-covariance constraints to describe covariance and autocorrelation characteristics. Expected calendar effects can be optimally arbitraged by controlling the memory or autocorrelation characteristics of the efficient portfolios. The Digital Portfolio Theory optimization model is compared to the Modern Portfolio Theory model and is used to find efficient portfolios with zero calendar risk for selected periods.

Key words: portfolio optimization, portfolio theory, digital signal processing, calendar anomalies

1. Introduction

The purpose of this study is to formulate the portfolio selection model to allow control of periodic risk factors using digital signal processing techniques. As with many digital signal-processing applications the payoff is substantial. The Digital Portfolio Theory model not only allows portfolio autocorrelation characteristics to be controlled in a one period model, but permits high definition of portfolio risk characteristics selected from very large universes. Financial return signals generated by stochastic processes will be described not only by their mean and variance, but also by their autocorrelation using a frequency domain representation. By describing finite sequences of returns as digital signals, an expression for the portfolio variance is derived that depends on unconditional autocovariance information in addition to unconditional variances and covariances. The Digital Portfolio Theory model uses the discrete-time signal processing definition of portfolio variance to derive a set of linear variance constraints to the Markowitz Modern Portfolio Theory return maximization model. The Digital Portfolio Theory optimization problem manages the risk characteristics of the optimal portfolio by controlling calendar and non-calendar, systematic and unsystematic components of the portfolio's risk. Controlling the periodic risk structure of the selected portfolio has become more important as a result of growing awareness of periodic calendar anomalies. The information set of the investor at the time portfolio decisions are made now includes estimates of calendar anomalies at weekly, monthly, quarterly, yearly and 4-year turning points. These calendar anomalies describe the stationary periodic timing elements of the market macrostructure.

The literature on these anomalies is large and growing. Linn and Lockwood (1988) and Hensel and Ziemba (1996) examined a monthly effect with greater return in the first half of the month. Beller and Nofsinger (1998) examined differences in monthly volatilities. Penman (1987) found a quarterly effect not related to firm size with higher returns in the first month of the quarter. Wachtel (1942) discovered a summer effect with bull market tendencies in summer months. The turn-of-the-year effect was found to be primarily a small firm effect by Rozeff and Kinnery (1976) and Keim (1983). Booth and Booth (1999) have measured an election effect with returns twice as large in the second two years of the presidential term and larger for small firms. Additionally, some investors believe that periodic effects may exist for much longer periods of time, such as 30, 40 or 60 years. Return series with periodic anomalies are not memoryless or random walks but are non-memoryless with non-zero autocorrelation. These processes may be at least weakly stationary. While return volatility processes are covariance non-stationary for short time periods, longer run periods display features of covariance stationarity, i.e. have variance and autocovariance independent of real time.

Studies in mean reversion have found 8-year mean reversion (see Poterba and Summers, 1988) and 4-year mean reversion (see Fama and French, 1988; Thorley, 1995). Fama and French (1988) find that a tendency for negative autocorrelation of long horizon return is always observed, although with less significance after 1940. Fama (1991) finds that variance tests provide weak statistical evidence against the hypothesis of no autocorrelation (random walk) despite low power resulting from short sample lengths.

There is growing recognition that long memory dependence is an inherent feature of return processes generated by financial systems. See, for example studies by Baillie (1996) and Bollerslev and Mikkelsen (1996). Andersen and Bollerslev (1997) utilize spectral analysis to measure long memory because of its ability to resolve the stationary components of long-period variance.

Models for testing long memory or persistence in a process often test for a slowly decaying autocorrelation function assuming the process is weakly stationary. Breidt, Crato, and de Lima (1998) present evidence indicating that the stationary long memory model provides an alternative to non-stationary volatility modeling. While market microstructure studies find evidence of covariance non-stationarity of short-run time-varying volatility processes, longer memory studies (Bollerslev and Engle, 1993) suggest that longer run return processes may be weakly stationary. Long memory may be the result of inherent properties of the return generating process rather than a result of external shocks inducing structural shifts in the volatility process.

Because long memory processes can be treated as weakly stationary, spectral estimators are increasingly being used in the analysis of long memory in the variance of stock returns (see Breidt, Crato, and de Lima, 1998). Long memory fractionally integrated processes are characterized by positive autocorrelation functions with hyperbolic decay and variance spectral density dominated by long periods, or low frequencies (see Baillie, 1996). Granger and Ding (1996) find evidence of long memory using the spectral density of absolute returns.

Digital Portfolio Theory controls the solution portfolio's sensitivity to calendar anomalies by controlling the portfolio calendar variance components. Regardless of the real or imagined existence of these anomalies, investors who believe calendar anomalies exist or believe returns do not follow a random walk, require a portfolio selection model which will adjust portfolio risk to correspond to their expectations and desired risk exposure. In addition to allowing control of the autocorrelation of the portfolio return process, the Digital Portfolio Theory formulation permits the mean-variance efficient set to be closely approximated, using a linear programming (LP) solution. The model can dramatically improve the array of available portfolio risk positions and allow timing considerations to be included in the portfolio decision by selecting efficient portfolios with risk characteristics matching investor calendar expectations and holding periods.

Section 2 reviews the historical development and basic properties of discrete signal processing. Section 3 describes Modern Portfolio Theory and previously developed algorithms. Section 4 applies signal-processing concepts to return sequences. Section 5 discusses the natural frequencies of financial signals. Section 6 explains the relationship between autocorrelation, variance spectral density and memory. Section 7 derives the variance of the portfolio return signal by adding systematic and unsystematic components of periodic risk. Section 8 derives the Digital Portfolio Theory portfolio selection optimization model. Section 9 discusses the criterion of utility maximization versus arbitrage in Digital Portfolio Theory. Section 10 examines the concept of time conditioning of the optimal portfolios. Section 11 looks at the question of a multiple period portfolio selection model. Section 12 describes the estimation of the Digital Portfolio Theory model parameters using short time series signal processing techniques. The performance of Digital Portfolio Theory for a small universe of securities is examined in Section 13.

2. Continuous versus Discrete-Time Fourier Analysis

One important way to model the stationary periodic risk structure of a return process is to decompose the variance using digital signal processing. How is it possible that a digital revolution is firmly established in the market place today and yet signal-processing techniques are not routinely applied to financial signals? The historical development of the discrete Fourier transform required to examine the spectral density function is important in order to appreciate the implementation of digital signal processing to financial signals. In the '20s and '30s continuous-time Fourier analysis was used to study electrical circuits and techniques of discrete-time Fourier analysis were developed to solve problems in numerical analysis and time series analysis. In 1930 Wiener published the first formal statistical treatment of signal processing using continuous-time Fourier analysis to define the variance spectral density at a continuum of frequencies. The increased use of digital computers in the '40s and '50s encouraged developments in discrete-time signal processing methods in engineering. However, by the '50s and '60s the calculation of continuous or discrete Fourier transforms was a prohibitive computational burden using slow speed and high cost mainframe computers.

In the mid '60s an algorithm know as the FFT (Fast Fourier Transform) was developed, greatly accelerating the pace of digital signal processing and applications. The digital FFT algorithm (Blackman and Tukey, 1959) was inherently a discrete-time concept. The FFT reduced the computational time for discrete Fourier transforms by orders of magnitude. The number of operations (adds and multiplies) was reduced from N^2 to N adds and $N/2$ multiplies. Much of the subsequent development in digital signal processing and the digital revolution reflects the nature of the FFT algorithm. Finite series as opposed to infinite series in continuous time were used to describe signals. In addition, signals were restricted to 2^n periods in length in order to realize the FFT's substantial computational efficiency. The transition from 16, to 32, to 64, to 128-bit architectures in the PC and other digital hardware and software is an artifact of the FFT. Ironically, the FFT algorithm has been largely responsible for the slow growth of the application of digital signal processing to financial signals. Financial signals require lengths of 3, 6, 12, 24, 48, etc. in order to capture longer-term calendar risk. The high precision offered by high-resolution digital signal processing methods depends on sampling at signal lengths that correspond to the natural frequencies of the financial system. Even today economic statistical software packages are often restricted to signal lengths of 2^n (4, 8, 16, 32, 64, 128, etc.) for computing spectral estimators.

In the 1970s Granger and Morgenstern (1970) and Praetz (1979) were among the first researchers to apply the FFT algorithm to continuous financial signals. At the same time, by the mid '70s the engineering community had derived a comprehensive theory of statistical digital linear signal processing systems and had evolved as a set of exact properties and mathematics applicable to the discrete-time domain. Digital signal processing systems theory utilizes more powerful mathematics than is available for continuous-time signals. For example, the integral required for the convolution of two signals in continuous-time becomes a simple summation in discrete-time because addition and subtraction for digital signals replace integration and differentiation in continuous-time. This property in conjunction with very high speed is responsible for today's digital revolution. In the '80s and '90s new techniques for high resolution finite discrete statistical signal processing, particularly for short data sequences, have become available. These new techniques are suited to the analysis of speech signals as well as stochastic digital financial signals (see Marple, 1987). Within the past five years, the greater speed of processors allows discrete Fourier transforms to be computed without using the FFT algorithm and as a result, financial researchers and practitioners are no longer constrained to signal lengths of 2^n. Today Fourier transforms can be directly computed for signal lengths of $3 * n$ using monthly returns, these lengths can be used to examine the calendar variance components of the market's macrostructure.[1]

2.1. PROPERTIES OF DISCRETE FOURIER TRANSFORMS

Statistical digital signal-processing involves the recovery of information from a signal that is embedded in noise. This is accomplished by repeatedly applying the discrete Fourier transform to the random signal, resulting in the variance spectral density of the process. In order to appreciate the usefulness of the discrete Fourier transform to financial signal processing, two important properties should be kept in mind. First, the discrete Fourier transform possesses the property of complete reciprocity. In other words, from the discrete Fourier transform of a time series of length N, the original series can be uniquely returned using the inverse discrete Fourier transform. Secondly, there is no implication that the original series is or must be periodic in order to compute the discrete Fourier transform. The discrete Fourier transform it is a non-parametric mapping from the time domain to the frequency domain.

2.2. THE UNIQUE NATURE OF FINANCIAL SIGNALS

Financial signal processing has been limited in development primarily because financial empirical researchers have not taken advantage of digital signal processing techniques. Additionally, progress has been impeded by the complexity that will be required in specialized algorithms needed for high resolution of financial digital signals. Like many digital signal processing applications that are characterized by complex algorithms, financial signal processing would not be possible using continuous-time signals. Speech processing, digital TV, digital audio, image enhancement, pattern recognition, digital telecommunication systems, data compression, geophysics and meteorological modeling all utilize digital algorithms. Greatly improved performance, low noise, high resolution, high accuracy, and high dependability characterize these digital systems.

Digital processing is having a dramatic impact on speech processing. It is interesting to note that financial signals are in several aspects similar to speech signals in that both speech signals and financial signals do not have the problem of random frequencies. Because of the shape of the vocal cavity, speech signals have known frequencies. Many financial market participants believe that financial markets have known frequencies based on the institutional nature of the markets and their calendar dependence. For example, one economic reason for annual dependence in financial signals could be that firm's fiscal year ending dates are not distributed with equal probably across all calender dates. While speech signals are non-stationary and time-varying, they can be treated as stationary for short periods (quasi-stationary analysis). Speech signals are stationary for periods up to half a second while financial signals display characteristics of stationarity for periods of a month and longer. Financial signals and speech signals differ in terms of their sampling rates. Speech signals are sampled in milliseconds while the stationary components of financial signals must be sampled monthly. Noise reduction techniques for short signals being developed for speech processing or pattern recognition may be adapted to financial signals. Digital image processing has made significant advances as a result of various techniques such as artificial intelligence and optimization to achieve image restoration and enhancement in the presence of nonlinear noise. The Digital Portfolio Theory utilizes discrete mathematics in conjunction with an optimization model to find desired portfolios of financial signals from very large universes of security signals.

3. The Markowitz Modern Portfolio Theory Portfolio Selection Model

The mean-variance problem originally formulated by Markowitz (1952) can be written either as maximization or a minimization problem. The maximization problem is the preferred formulation for the investor since it is intuitively more appealing to maximize return and constrain risk. The maximization formulation is convenient for adding constraints and/or integer variables, which was difficult using the risk minimization problem. The maximizing formulation also allows other linear functions such as proportional transactions costs to be added in the objective function and fixed costs to be imposed using integer variables. The fact that most financial problems can be represented using a maximum flow network structure that is constrained by flow conservation has been recognized by Crum, Klingman and Travis (1979), Mulvey (1987), Mulvey and Vladimirou (1991), Jones (1992), and Dantzig and Infanger (1993). In Equations (1) and (2) the Modern Portfolio Theory model assumes that autocorrelations are zero or are of no importance to the investor.

Maximize

$$E(\tilde{r}_p(t)) = \sum_{j=1}^{N} W_j E_j(\tilde{r}(t)) = \sum_{j=1}^{N} W_j \mu_j \tag{1}$$

subject to

$$\mathrm{var}(\tilde{r}_p(t)) = \sum_{i=1}^{N} \sum_{j=1}^{N} W_i W_j \mathrm{cov}(\tilde{r}_i(t), \tilde{r}_j(t)) \le c \tag{2}$$

$$\sum_{j=1}^{N} W_j = 1\,,$$

where W_j = the fraction invested in security j; $\tilde{r}_p(t)$ = stochastic return on the portfolio in period t; N = number of securities in the universe, $j = 1, 2, \ldots N$; c = right-hand side (RHS) constant.

Implementation of Modern Portfolio Theory involves two interdependent problems. The first is the data representation problem. The covariance matrix (2) has been the biggest difficulty encountered in attempts to find large scale mean-variance efficient optimizations because of the density of non-linear terms. The second problem is the computational procedure or non-linear programming algorithm needed to calculate efficient portfolios. Significant advances are now being made in non-linear programming techniques using interior point methods.

Previous mean-variance algorithms can be categorized into two broad approaches: those that minimize variance and those that maximize expected return. Each of these categories can be further subdivided into algorithms that give exact solutions to the mean-variance efficient set and algorithms that give approximate solutions. Exact algorithms that minimize variance include the Markowitz (1956), Markowitz, Todd, Xu, Yamane (1992) critical line method, the Wolfe (1959) simplex method for quadratic programming, and the Alexander (1976) algorithm using the Lemke (1965) complementary pivot solution.

Variance minimization algorithms that give approximate solutions have been developed to accommodate large-scale problems. These techniques simplify the covariance matrix using index or factor methods. The Sharpe (1963) diagonal method uses a single index to simplify the covariance matrix. Other researchers have proposed multi-index or multi-factor models that are similar in concept to the single index model but try to capture some unsystematic influences while taking advantage of a sparse covariance matrix (e.g., Cohen and Pogue, 1967; Elton and Gruber, 1973; Rosenberg, 1974; Pang, 1980; Markowitz and Perold, 1981; Perold, 1984).

Elton, Gruber and Padberg's (1976) single index model for finding the mean-variance efficient portfolios is popular because it does not require quadratic programming or an LP approximation. This approach, however, does not lend itself well to the addition of constraints commonly used in portfolio management. For example, to add upper bounds on the decision variables presents a non-trivial problem (see Elton, Gruber and Padberg, 1977).

Algorithms that maximize return subject to a quadratic constraint set utilize substitute linear constraints. Sharpe (1971) developed an approximate algorithm that used a piece-wise linear approximation to the covariance matrix. Glover and Jones (1988) developed a dual feasible direction algorithm that gives the exact solution using a cross-covariance constraint set. In practice software packages that solve for efficient mean-variance portfolios from large universes are still not readily available to the researcher or the investor. Digital Portfolio Theory offers a linear model that will allow efficient portfolios to be found using conventional LP packages. This model is more intuitively understandable than the original Modern Portfolio Theory model.

4. The Digital Signal Processing Model

Stochastic digital signal processing is concerned with representation, transformation, manipulation and estimation of signals and the information they contain. Digital Portfolio Theory assumes that return processes are locally wide sense stationary and moments can be calculated by taking time series averages (ergodic). Essential to digital technology is discrete measurement of finite length processes. Digital signal processing represents return sequences with finite series as opposed to the infinite series required in continuous time. Time series security returns are considered digital signals since they are constructed from price data at discrete-time intervals, such as daily, weekly, or monthly, etc. Digital signal processing transforms finite length discrete-time series sequences into the sum of periodic exponential functions with the period of each term of the summation decreasing harmonically in length. The discrete Fourier Theorem states (see Jenkins and Watts, 1969) that any finite discrete stationary stochastic process can be described non-parametrically by a finite sum of complex exponentials. In sine form a digital return process can be represented as follows:

$$\tilde{r}_j[n] = \mu_j + \sum_{k=1}^{K} R_{kj} \sin(k\omega n + \tilde{\theta}_{kj}) \quad n = 1, 2, 3, \ldots, T\,, \tag{3}$$

where $\tilde{r}_j[n]$ = stochastic return of security j in period $n\delta t$; n = an integer indicating the nth place in the finite sequence; μ_j = expected return for security j; R_{kj} = amplitude of the kth periodic term for signal j; θ_{kj} = kth random phase of the jth return signal; ω = angular frequency of the longest period sampled $\omega = 2\pi/T$; K = number of harmonic period term ($K = T/2\delta t$); T = length of the digital signal and 1st ($k = 1$) harmonic; δt = time interval between samples.

The Fourier series, Equation (3) may be written in exponential form, in sine form, in cosine form, or as the sum of sine and cosine terms. The sine form is useful in this context since it is convenient for presenting the addition of digital financial signals using traditional vector or phasor algebra. Here K, R_{kj} and ω (omega) are constants and the $\tilde{\theta}_{kj}$ are independent random variables. The amplitudes, R_{kj}, have units of standard deviation (risk). The brackets [ ] indicate a digital process. The probability distributions of the phase angles, $\tilde{\theta}_k$, are uniformly distributed over (0,2π). Since the phase angles are random variables, Equation (3) cannot be used to solve for the returns. The K sine functions form a complete orthogonal set. Each of the K terms in (3) has mean zero and is uncorrelated to other harmonic terms. Equation (3) has the following characteristics:

$$E(\tilde{r}_j[n]) = \mu_j \tag{4}$$

$$\mathrm{var}(\tilde{r}_j[n]) = \frac{1}{2}\sum_{k=1}^{K} R_{kj}^2\,. \tag{5}$$

The variance of the digital return process in Equation (5) is not dependent on $\tilde{\theta}_k$. The variance of the return signal (5) is made up of the sum of K variance terms, R_k^2. Following Modern Portfolio Theory, Digital Portfolio Theory will utilize this variance as the absolute measure of risk of the return process. The estimates of the squared amplitudes, R_k^2, describe the variance spectral density or periodic volatility pattern of a financial signal. The variance spectral density gives the same information as the autocorrelation function in the time domain. The relative amplitudes measure the presence of autocorrelation. For example, when the variance of a security's digital return signal has a large yearly component, the amplitude value, R_k, corresponding to a yearly variance will be larger (or peak) relative to other amplitudes. When a digital return signal follows a random walk, all R_k values will be equal, or insignificantly different, the autocorrelation at all lags will be zero and the process is called memoryless.

The K terms in the summations in (3) and (5) are called harmonics since the lengths of their periods decrease harmonically. The number of periods, K, is equal to one half the signal length, T. The length of the kth harmonic period, p_k, is given by Equation (6).

$$p_k = \frac{T\delta\tau}{k} \quad k = 1, 2, \ldots, K = T/2\delta t\,. \tag{6}$$

The frequency is inversely proportional to the period length. There are K periods and K discrete periodic variance components that make up the total variance in (5). Suppose T is chosen as 48 months with a sampling interval of one month, then $k = 1$ gives a four year risk component, $k = 4$ gives a one year risk term, and $k = 16$ gives a quarterly (three month) risk term, etc. The period of the first harmonic ($k = 1$) is T, the signal length. Statistical digital signal processing of return signals estimates the values of the amplitudes, R_k, by averaging discrete Fourier transforms over many repetitions of the signal. The result is called the variance spectral density or risk spectrum of the stochastic process. Digital Portfolio Theory describes asset risk by these K variance components corresponding to the K unique periods, or frequencies in the digital financial signal's volatility.

The cross-covariance or cross-correlation describes the relationship between two return signals. The covariance can be written as the sum of K cross-covariances between security i and security j.

$$\mathrm{cov}(\tilde{r}_i[n], \tilde{r}_j[n]) = \frac{1}{2}\sum_{k=1}^{K} R_{ki}R_{kj}\cos(\theta_{kij})\,, \tag{7}$$

where θ_{kij} = the kth phase-shift between security i and j.

The phase-shift gives the lead or lag between two processes at the same frequency. The cross-correlation, ρ_{kij} (rho), between the kth periodic component of the processes of security i and security j is equal to the cosine of the phase shift (see Wiener, 1970).

$$\rho_k(\tilde{r}_i[n], \tilde{r}_j[n]) = \cos(\theta_{kij}) . \tag{8}$$

The phase-shift corresponds to correlation. Like correlation, phase-shift is not a random variable. The phase-shifts between two random processes are measured using cross spectral analysis. There is always zero cross-correlation between two random processes at different periods or frequencies since they are orthogonal.

5. Resonate Frequencies of Financial Signals: Calendar versus Non-Calendar Risk

Resonate or carrier frequencies may result from the institutional and cultural calendar structure of the markets. To benefit from high resolution digital techniques, the sampling interval and the signal length must correspond to this periodic calendar structure. While news arrivals may influence different assets in different ways, they may have been generated by factors related to the overall economy, inducing a covariance stationary process. The interaction of a large number of regularly scheduled macroeconomic announcements may result in seasonal and long run calendar memory or autocorrelation. This section will show that the K terms of the variance spectral density function in Equation (5) can be broken into calendar and non-calendar risk components. In Equation (6) the selection of δt, the sampling interval, plays an important role in determining the measured period length and frequency. For example, one difficulty encountered with financial signals is that the number of weeks per month and days per month is not integer. As a result, weekly or daily sampling will not allow the resolution of important calendar periods since 52 or 260 are not divisible by 12. Most studies using spectral estimation have found the low frequencies, or longer periods, dominate in explaining the variance of financial signals. For instance, Cooper (1974) and Durlauf (1991) found periods of one month and longer explain the largest amount of return variance. To describe longer term variance characteristics financial signals should be sampled monthly with signal lengths of 48 or 96 months. Since the 96-month signal may violate the assumption of local stationarity in the signal processing model, the 48-month signal can be used as the fundamental resonate frequency for financial return processes generated in the market macrostructure. Table I gives the discrete harmonic period lengths from Equation (6) for the 4-year signal sampled monthly.

With a 48-month signal length there are 24 periodic variance terms in Equation (5). These can be separated into periods that represent meaningful calendar-time intervals and those that do not. The calendar risk components correspond to institutionally relevant time periods. In this example, there are five calendar risk factors. Shorter or longer calendar risk components might be measured, but these five variance factors will still persist. An empirical study is not necessary to identify the length of the calendar risk factors since they are common knowledge to all

Table I. Calendar and non-calendar risk components with signal length of $T = 48$ months. The unique periodic risk factors of a four-year signal sampled monthly.

k	Period p_k (months)	Calendar risk factor	k	Period p_k (months)	Calendar risk factor
1	48.0	4 year	13	3.7	
2	24.0	2 year	14	3.4	
3	16.0		15	3.2	
4	12.0	1 year	16	3.0	$\frac{1}{4}$ year
5	9.6		17	2.8	
6	8.0		18	2.7	
7	6.9		19	2.5	
8	6.0	$\frac{1}{2}$ year	20	2.4	
9	5.3		21	2.3	
10	4.8		22	2.2	
11	4.4		23	2.1	
12	4.0		24	2.0	

investors. These periodic calendar risk components are the resonate frequencies for all financial signals.

6. Autocorrelation and the Periodic Dimensions of Memory

While most economists are familiar with the autocorrelation function, the spectral density function gives the equivalent information but focuses on a different aspect of the nature of the time series while both describe the memory of the process. There is an inverse relationship between the time and the frequency domains. The autocorrelation function reflects the association between successive values of a series while the spectral density describes how the variance of the random process is distributed with frequency. Both represent the second moments of the process. The autocorrelation function is described by the relationship between all explicit time lags. If there is no autocorrelation in a series and therefore no memory, the autocorrelation at all lags will be zero in the time domain. On the other hand, in the frequency domain, the series with zero autocorrelation or no memory will have a variance spectral density that is equally distributed at all frequencies.

A series with positive autocorrelation will result in a spectrum with large variance at low frequency, or long periods. Conversely, a series with negative autocorrelation will result in large variance at high frequency. A series that is periodic will have an autocorrelation function that is also periodic with positive and negative values at different lags. At the same time, the variance spectrum will have a peak corresponding to the frequency of the periodicity (see Jenkins and Watts, 1969). In the frequency domain the periodic variance spectrum described by Equation (5) defines the autocorrelation function. Since this description focuses on the stationary components of risk at time interval differences, it is particularly appropriate for the portfolio risk diversification problem. The computation of portfolio variance using the autocorrelation function would be prohibitively complex in the time domain but is possible in the frequency domain because the Fourier transform is type invariant under addition. This allows periodic variance components for each asset to be added to find the corresponding periodic portfolio risk characteristics. The efficient digital mathematics used to combine variance components into portfolios is described in the next section.

While there has been considerable research on describing short term time-varying risk using ARIMA or GARCH models, digital signal processing allows the inclusion of stationary components of long memory or stationary risk factors related to calendar anomalies or periodic announcement effects. Calendar anomalies have been found to exist for as long as data has been collected. Methods such as GARCH, wavelets, or atomic decomposition attempt to forecast the time location and frequency characteristics of non-stationary and non-linear return generating processes. These methods are important in signal analysis whenever transient behavior or discontinuities dominate a signal. For example, wavelet analysis can be used to describe a signal that is localized in time. Spectral estimators offer no information about the time location of the stochastic process volatility, but measure only stationary risk characteristics of the overall volatility process. Consequently both Digital Portfolio Theory and Modern Portfolio Theory utilizes portfolio variance to achieve diversification, they are not prediction models.

7. Signal Processing Measurement of Risk and the Covariance Matrix

Risk in Digital Portfolio Theory is defined by K independent periodic components of the total variance. The units of risk are the same as those in Modern Portfolio Theory. Variance components are periodically related and sum to the total variance. This section derives the portfolio variance when asset return signals are generated by Equation (3). To compute the portfolio variance, the portfolio amplitudes, R_{kp}, for each of the K risk components in Equation (5) must be computed. In order to add the return signals of individual securities, their phase-shifts must be estimated. To find the portfolio autocorrelation structure in the frequency domain the phase shifts, θ_{kmj}, or lead or lag relationship between all securities must be measured. The best way to measure phase shifts for each security is to measure them relative to a reference series or signal. Analogous to the single index method, the K phase-shifts, θ_{kmj}, for the jth security's signal can be measured relative to an index return signal. The index return process may or may not be generated by an efficient market portfolio. In order to find portfolio variance, the complex addition of digital signals

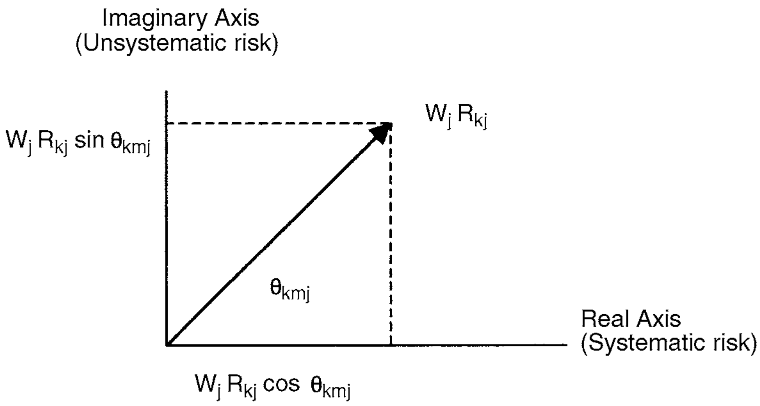

Figure 1. Complex addition of risk phasors to find R_{kp}. To compute the portfolio risk the periodic risk phasors can be added across securities by adding imaginary and real components separately. The cosine terms represent systematic risk relative to a reference index since the in-phase components will have phase-shift, θ_{kmj} equal to zero or 180 degrees. The sine terms represent unsystematic risk since the out-of-phase components will have phase-shift, θ_{kmj} equal to 90 or 270 degrees.

can be used. To find the portfolio amplitudes, R_{kp}, it is convenient to introduce the concept of a risk phasor or rotating vector as shown in Figure 1. The risk phasor has magnitude and direction. The magnitude of the risk phasor for security j for periodic risk factor k is its amplitude, R_{kj}, or standard deviation for the kth risk component. The direction of the risk phasor for security j for risk factor k is the phase-shift, θ_{kmj}, between the return signal of security j and a reference signal m. Phase-shift does not give any time location information but measures relative covariance information.

To find the portfolio variance, the weighted risk phasors must be added. This can be accomplished by replacing the K random variables, $\tilde{\theta}_{kj}$, in Equation (3) with the K phase-shifts, θ_{kmj}. Since only the phase-shift is important for diversification the $k\omega n$ in Equation (3) is dropped. Since the K phasors for each security are orthogonal, they can be added for each period or frequency separately. By definition the sine term in Figure 1 is the imaginary part of the risk phasor. The cosine term is the real part. When the phase-shift is measured relative to a reference index, the cosine terms represent the amount of portfolio variance explained by systematic risk with respect to that index. If the phase-shift is zero (perfectly in phase), the cosine is one. The cosine or real part of the securities return process is in phase with the market index representing systematic risk. When the phase shift is 180 degrees, the cosine is minus one, representing negative systematic risk. In the Digital Portfolio Theory model systematic risk can be both positive and negative representing high positive or negative cross correlation (Equation (8)). In the same way, the sine terms represent unsystematic risk since the sine of 90 degrees equals one.

Since the systematic and unsystematic parts of the risk phasor are at right angles, they can be added. The resultant portfolio amplitude, R_{kp}, is found by applying the Pythagorean Theorem.

$$R_{kp} = \left(\left(\sum_{j=1}^{N} W_j R_{kj} \cos\theta_{kmj} \right)^2 + \left(\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj} \right)^2 \right)^{\frac{1}{2}} . \tag{9}$$

Using Equation (5) the variance of the resultant portfolio return signal can be written:

$$\operatorname{var}(\tilde{r}_p[n]) = \frac{1}{2} \sum_{k=1}^{K} \left(\left(\sum_{j=1}^{N} W_j R_{kj} \cos\theta_{kmj} \right)^2 + \left(\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj} \right)^2 \right) \tag{10}$$

$$\text{Portfolio risk } = \frac{1}{2} \sum_{k=1}^{K} ((\text{systematic risk}_{pk})^2 + (\text{unsystematic risk}_{pk})^2) .$$

Since the variance (10) of the portfolio return signal is assumed locally stationary it is not time dependent. The derivation of portfolio variance did not assume any distribution for the underlying security return processes. Consequently the distribution of the security returns may be skewed or non-normal. Furthermore the security return processes may be random walks or may be non-memoryless with non-zero autocorrelations.

The risk phasor allows systematic and unsystematic risk relative to a reference signal to be defined and measured in a more precise way than was available using covariance measures. The definitions of systematic and unsystematic risk in this context are related to the index being used to measure phase-shift. For example, suppose an investor wishes to profit from the January effect. This investor may want to increase exposure to yearly risk but since the January effect is primarily a systematic market phenomenon it is the systematic yearly risk relative to a market index that the investor should be exposed to. A particular security may have a high level of yearly risk but if most of this risk is not systematic the security would not be useful for January effect arbitrage.

8. The Digital Portfolio Theory Model

In this section the periodic representation of portfolio variance is used to develop a substitute constraint set for the traditional variance constrained mean-variance model. Since each of the K pairs of cosine and sine summations in (10) are orthogonal or uncorrelated, local efficiency must be reached with respect to each to find the efficient portfolio. The portfolio variance constraint (2) can therefore be written as K independent cross-covariance constraints:

$$\left(\sum_{j=1}^{N} W_j R_{kj} \cos\theta_{kmj}\right)^2 + \left(\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj}\right)^2 \leq b_k \quad \text{for } k = 1, 2, 3, \ldots, K \,. \tag{11}$$

The kth periodic component of portfolio variance will have an efficient frontier in periodic risk-return space that is independent of the other components. The efficient frontier in the kth risk-return dimension consists of the undominated portfolios in that plane. The K constraints can be added as separate constraints to the return maximization problem. These periodic constraints are still non-linear but are independent and have a more symmetric form than the expression for the covariance matrix given in Equation (2). The mathematical properties of the discrete-time Fourier transform provide a highly efficient means of computing and controlling periodic components of the portfolio variance that cannot be achieved for continuous-time processes. The symmetry allows a set of substitute constraints to replace each of the cross-covariance constraints in Equation (11).

Constraining the systematic and unsystematic variance terms separately creates a relaxed constraint set. In the derivation of the variance of the portfolio's return signal in Figure 1, these terms were right-angle projections. To effectively constrain the square of these terms, the systematic and unsystematic terms in (11) can each be constrained to be less than a constant right-hand side, b_k, and greater than the negative of the same constant. The relaxed LP signal processing portfolio selection model is:

Maximize

$$E(\tilde{r}_p(t)) = \sum_{j=1}^{N} W_j E(\tilde{r}_j(t)) = \sum_{j=1}^{N} W_j \mu_j \,. \tag{12}$$

Subject to $4K$ constraints; $k = 1, 2, 3, \ldots, K$

$$\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj} \leq b_{sk} \tag{13}$$

$$\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj} \geq -b_{sk} \tag{14}$$

$$\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj} \leq b_{uk} \tag{15}$$

$$\sum_{j=1}^{N} W_j R_{kj} \sin\theta_{kmj} \geq -b_{sk} \tag{16}$$

$$\sum_{j=1}^{N} W_j = 1 \tag{17}$$

$$W_j \geq 0 \qquad j = 1, 2, 3, \ldots, N\,, \tag{18}$$

where $b_{sk} = k$th systematic RHS; $b_{uk} = k$th unsystematic RHS.

Equations (12)–(18) give the Digital Portfolio Theory model formulation. Note that it is a completely linear model and allows much greater control over the components of portfolio variance. For each of the 24 periodic risk factors shown in Table I there are 4 constraints resulting in 96 constraints to control portfolio risk. These constraints allow the optimal portfolio allocation given the investors desired exposure to each periodic variance contribution to the total portfolio variance. By choosing appropriate values of the constants, b_{sk} and b_{uk} , the signal processing model allows diversification to be applied independently to the different periodic, systematic, and unsystematic risk components that make up the portfolio variance. This gives more control over the characteristics of the resultant efficient portfolios selected. Additionally, because of the relative amplitudes, R_{kp}, of the solution portfolio reflect the autocorrelation structure, memory characteristics of the efficient portfolios can be controlled.

Because there are effectively K independent efficient frontiers a particular investor may select an efficient portfolios that has a high risk in the kth dimension and simultaneously has low risk in the k+1 risk dimension. Another investor could have some other combination of periodic risk preferences and therefore a quite different efficient portfolio would be found using Digital Portfolio Theory. Two investors with the same total risk tolerance may hold very different portfolios. Each of the K efficient frontiers will be independent of investor preferences. In general a particular investor's optimal portfolio may not be the same as a representative mean-variance investor or on the mean-variance efficient frontier. The collection of efficient portfolios for different investor preferences will be bounded in mean-variance space by the mean-variance efficient set. The right-hand-sides, b_{sk} and b_{uk}, in Equations (13)–(16) can be adjusted to give different efficient portfolios, depending on the individual's desired exposure to risk for systematic or unsystematic components of the portfolio's variance. By setting $b_{sk} = b_{uk} = 0$ for a particular value of k, efficient portfolios can be found with no risk do to the kth periodic risk component.

An investor may be willing to accept a large amount of portfolio variance from low frequency components (long time differences) and a small amount of variance from high frequency components (short time differences) or vice versa. For example, a particular investor may be willing to bear 4-year risk but not quarterly risk. Suppose an investor must buy a portfolio today. Additionally, suppose that the investor believes that security returns may be adversely affected by quarterly earnings announcements one month from today. The investor can solve for the efficient portfolio with no systematic and unsystematic quarterly risk using Digital Portfolio Theory. This portfolio will be made of securities whose return signals are not sensitive to the risk associated with the quarterly earnings announcements, therefore, the investor will be protected based on a forecast of quarterly earnings below expectations. At the same time the investor may anticipate a four-year election cycle with higher returns in the last 2 years. If his or her holding period will include the second half of the election term, the investor may prefer efficient portfolios with higher 4-year risk. Since this hypothetical election cycle anomaly is systematic, this investor may not want to be exposed to 4-year unsystematic risk.

In Equation (18) the W_j variables are constrained to be non-negative. Short selling can be added by dropping the non-negativity constraint. Allowing short selling, efficient portfolios could be selected from a given universe of securities that have only quarterly, or only yearly risk, etc. Jones (1992) has used these pure periodic disturbance portfolios in an Arbitrage Pricing Theory (APT) model. The maximization LP formulation of Digital Portfolio Theory allows the use of integer variables to model fixed commission cost and other fixed costs. Additional linear constraints can be added to control fundamental factors such as growth in earnings per share, P/E, market capitalization, etc. in the solution portfolio.

The Digital Portfolio Theory model (12) with no short selling will find the same efficient portfolios using an efficient index or an inefficient index. The reference index need not be an efficient portfolio to find the efficient portfolios. If the index is not a market index then systematic and unsystematic risk are measured relative to this reference index. If the index used is a market portfolio, an approximate capital market line (CML) can be found by setting the $b_{uk} = 0$ for all k and solving the Digital Portfolio Theory model. As the number of securities in the universe approaches the universe of all securities in the market this approximate CML will approach the exact CML.

The solution to the signal processing portfolio selection model can be obtained using LP. The problem can be rapidly solved for very large universes. In Modern Portfolio Theory the quadratic covariance constraint in (2) contains $N(N+1)/2$ different terms. In Digital Portfolio Theory the linear relaxed constraint set in (13) to (16) contains $4KN$ terms. For example, for a universe of 20,000 securities the portfolio variance for the Modern Portfolio Theory model requires over 200 million terms. The Digital Portfolio Theory model requires less than 2 million terms with $K = 24$.

9. Autocorrelation, Utility, and Arbitrage

Because there are K unique independent efficient frontiers in K risk dimensions in Digital Portfolio Theory, the personal utility function in mean-variance space does not define the individuals optimal portfolio. While autocorrelation has not previously been described in the single period Modern Portfolio Theory model the effects of autocorrelation have been examined in the sequential investment problem when yields are serially correlated. In this case Harkansson (1971) used logarithmic utility functions to maximize the end of the investment horizon terminal wealth and postulated that this would induce single period utility functions which are myopic or independent of yields beyond the current period. If myopic utility is the case, investment horizon is irrelevant and risk aversion determines the optimal portfolio. A myopic decision strategy will treat every period decision as a signal period decision and require a utility function that allows the investment weights to be independent of wealth. Harkansson (1979) examined the situation when autocorrelation exists in yields and suggested that the only utility that maximizes the log of the end of period wealth results in myopic optimal decisions in each period. We know from the Digital Portfolio Theory model that if calander effects do exist, returns will have non-zero autocorrelations and that the single period portfolio decision will treat the components of periodic risk differently in different periods. It seems likely then that the utility functions in a particular periodic risk return subspace will not be the same from period to period. Additionally the trade off between the utility designated to each periodic risk component will change from period to period.

In the case of mean-variance autocorrelation optimization (Digital Portfolio Theory), it may be appropriate to replace the criterion of subjective expected utility maximization with the principle of no-arbitrage. The concept of arbitrage is clearly linked to market equilibrium. Specifically the arbitrage argument is that investment portfolios that require no net investment should not have a positive return. The arbitrage intuition is at the base of capital structure theory (Modigliani and Miller, 1958) the options pricing model, Black and Scholes (1973) and Merton (1973) and the arbitrage pricing of Ross (1976). Nau and McCardle (1991) suggest that the principle of no arbitrage is more fundamental than utility maximization and that the LP (Linear Programming) optimization problem that maximizes expected return is in fact the exploitation of arbitrage opportunities by rational agents. When we allow short selling in the Digital Portfolio Theory model by letting the weights in Equation (18) to be negative the solution includes short selling opportunities and will generally contain all securities in the universe. In the Digital Portfolio Theory model the K periodic variance constraints in conjunction with the return maximizing objective represents a no arbitrage condition with respect to the periodic components of the portfolio's stationary variance and cross covariance. A particular investor's optimal efficient portfolio will be uniquely defined by the investors desired exposure to the K independent risk components. The desired exposures to these components must dependent on factors such as the investor's holding period, purchase date and expectations about the existence of calendar effects as well as overall risk exposure. It may be reasonable that objective of utility maximization in Modern Portfolio Theory may be replaced by Arbitrage Pricing

Theory with periodic optimally arbitraged portfolios found using the single period Digital Portfolio Theory LP solution. Grinblatt and Titman (1987) suggest that exact arbitrage pricing is equivalent to local mean-variance efficiency with respect to a set of reference portfolios. These reference portfolios will be the K orthogonal efficient portfolios that have no risk with respect to the other $K - 1$ periodic risk components.

For example, suppose you are required to purchase an investment today and your holding period is one year. It is a one period portfolio decision. In Modern Portfolio Theory you determine your level of total risk exposure and compute your optimal efficient portfolio. In Digital Portfolio Theory you consider your expectation about cyclical effects over your holding period. If your holding period is going to include January you may increase your exposure to 1-year systematic risk. If you believe that your holding period will be during a bear market, you reduce your systematic risk relative to your unsystematic risk particularly for longer periodic components. Digital Portfolio Theory is not a multiple period portfolio decision model but the single period decision is based on timing considerations.

10. Conditional and Unconditional Efficiency

Conditional models allow expected returns and covariances to vary through time (Harvey, 1989). Time-varying risk models used to test the conditional CAPM (De Santis and Gerard, 1997) specify the dynamics of the conditional moments by allowing correlations among asset returns to change with market conditions, while assuming a covariance stationary process for the unconditional variance covariance matrix. In the case where traders use information available at the time of trading to condition historical movements, Hansen and Richard (1987) propose that there is both an unconditional mean-variance frontier and a conditional mean-variance frontier. The manner that estimates are modeled to vary through time generally depends on past short-term forecasting errors. The inclusion of long memory components of the variance is not included in conditioning. While Digital Portfolio Theory is an unconditional model, a conditional model could be formulated by conditioning the means and cross covariances.

Digital Portfolio Theory, because it includes autocorrelation in the one period model, adds a new prospective to the idea of a conditional efficient portfolio. Rather than condition on forecasting errors Digital Portfolio Theory allows conditioning on the date or time location. The conditional information that will determine the settings of the periodic constraints in (13) to (18) depends on the calendar date that trading will take place and the holding period of the investor. Suppose that we assume that processes are covariance and autocovariance stationary. Imagine that our trading date is in the middle of December, it is at the beginning of a 4-year election cycle and our holding period is two years. Additionally, suppose we believe that the quarterly, yearly and four year anomalies, as observed in the literature will persist. In this case we may want to increase our exposure to yearly risk and quarterly risk to benefit from the turn-of-the-year and beginning of the quarter effect but may want to reduce our 4-year risk exposure since it is early in the election cycle. On the other hand, suppose that our trading date will be the first of March in the second half of a 4-year election cycle and our holding period is three months. Now we have missed the turn-of-the-year effect and it is at the end of the quarter. We may want to reduce our yearly and quarterly risk and increase our 4-year and 2-year risk to benefit form securities with high longer-term variance contributions. The investor's trading date, holding period, and expectations about market direction and calendar effects will condition the unconditional Digital Portfolio Theory model.

11. The Multiple Period Problem

Digital Portfolio Theory is a single period model allowing control of the level of risk produced by various periodic variance components related to the autocorrelation. For example, the amount of 4-year variance in the solution portfolio can be controlled even when the investor's holding period is one month. In order to solve multiple period problems it is necessary to start with a comprehensive representation of the single period problem. The multi period model is complicated by the interrelationship of the mean, variance, covariance and autocorrelation in multiple periods. Intertemporal portfolio theory must consider the holding period length, time horizon, number of portfolio revisions, and intermediate consumption and liability streams. Multiple period theory must address the change in wealth over time of the portfolio and the effect of autocorrelation on multiple time related decisions. Expected autocorrelation characteristics may be important for the future investment decisions. The multiple period problem consist of a sequence of decision problems that are contingent upon outcomes in previous periods and on new information arriving. This problem is more complicated when autocorrelation is present in returns since decisions in earlier periods must take into account probability distributions in future periods.

Mossin (1968) and Chen, Jen and Zionts (1971) present a classic framework for the mean-variance multi period problem by using backward recursion and dynamic quadratic programming. Their conceptually appealing formulation however results in such a computational burden, particularly with transaction costs and large number of assets, that it has remained impractical. Other studies of the multiple period model include Smith (1967), Samuelson (1969), Merton (1969, 1972), Fama (1970), Hakansson (1971a, b), Elton and Gruber (1974a, b, 1975), Winkler and Barry (1975), Glover and Jones (1988), Mulvey and Vladimirou (1989), Dumas and Luciano (1991), Ostermark (1991), Dantzig and Infanger (1993), Grauer and Hakansson (1993), Gunthorpe and Levy (1994), Mulvey and Ziemba (1998), and Grinold (1999). Assuming autocorrelations are zero Li and Ng (2000) proposed an analytical expression for the multi period mean-variance efficient frontier but do not find an efficient solution methodology.

The purpose of this study is to present the single period model consistent with periodic autocorrelation using signal processing. Digital Portfolio Theory model is the first portfolio selection formulation capable of controlling components of autocorrelation in a one period model. A multiple period model that permits returns to have non-zero autocorrelatoin or assumes that investors have memory may extend Digital Portfolio Theory to multiple periods. Digital Portfolio Theory has several advantages to previous approaches to the multiple period problem. First the representation of risk in the Digital Portfolio Theory model has distinct mathematical advantages over previous formulations. Not only can risk be easily added across securities in a portfolio as demonstrated in this paper but the multiplication of stochastic risky return signals times stochastic monetary flows (convolution) that occurs from one period to the next can be facilitated using the new digital mathematics of linear systems theory. The multiple period stochastic maximum flow network model seems to offer the ideal framework for formulating the multiple period problem. Previous attempts to utilize the multi period network model have been restricted to scenario methods (see Mulvey and Vladimirou, 1991). The signal processing representation of risk allows the distributions encountered in the multiple period stochastic portfolio network models to be fully described.

The second advantage of Digital Portfolio Theory in it application to the multiple period problem is that specific trading dates can be related to perceived calendar anomalies and each period's risk can be adjusted accordingly. Some Investors may believe that some securities have returns that are temporally dependent in a predictable pattern. For example, investors who believe regular calendar effects exist. An optimal multiple period hedging strategy could be derived that would consist of holding portfolios to maximize terminal wealth subject to multi period periodic risk constraints. Suppose the investor will rebalance or trade every month, then based on the calendar month and the calendar year the investor's objective in terms of risk exposure to calendar and non-calendar variance factors will change every month and the solution portfolio will change. Because of the greater control over the portfolio variance using Digital Portfolio Theory, the composition of the selected portfolios will change faster than when the mean, variance and covariance are the only factors being considered.

12. Estimating the Digital Portfolio Theory Parameters

In order to test Digital Portfolio Theory the periodic variance components and phase-shifts for each security must be estimated. Twenty securities were chosen at random from the S&P 500 securities. The S&P 500 market returns were used as the index. The data was obtained from the Standard and Poor's COMPUSTAT database. Monthly returns over a 16-year period were used to compute the variance and cross covariance spectrums using a 48-month signal length. The Welch (1967) method is one of the most useful of the new high resolution, small sample, digital signal processing techniques for estimating the variance and cross covariance spectrum.[2] This method was used to estimate the K periodic risk amplitudes, R_k, and phase-shifts, θ_{kmj}. To implement the Welch method the sixteen years of monthly raw returns from June 1977 to May 1993 were divided into 7 overlapping segments of 48 months each. The 4-year signal length allows high resolution of the variance spectrum for the periods shown in Table I. Because financial return signals are masked to a large extent by noise, it is important that periods or frequencies are chosen to correspond to the institutional structure generating the return signals. A rectangular data window was used since it was anticipated that the return variance spectrums would be relatively flat. The Welch method averages the Fourier transform results to find the variance spectrum and the phase-shifts for each security relative to the reference or index return signal.[3]

12.1. WINDOWING FINANCIAL SIGNALS

Windowing is used to control the effects of sidelobes in the spectral variance estimators. Volatility estimates adjacent to peak or true frequencies can be biased. Leakage or aliasing to adjacent frequencies from peaks can mask or cancel out weaker signals. Side lobe level can be smoothed using alternative spectral windows but only at the cost of a reduction in spectral resolution. The rectangular window gives the narrowest resolution but has the highest sidelobes. The triangular window also called the Bartlett window and the squared cosine window called the Hanning window will result in fewer sidelobes with lower resolution, (see Harris, 1978; Parzen, 1957).

Window selection depends on weakness of the signal components being examined and the distance in frequencies between signal components. If there are strong components separated by large distances in frequencies the triangular or Hanning window should be used to diminish sidelobes. As spectrums become flatter the importance of using the triangular or Hanning windows to reduce sidelobes is diminished. The rectangular window is appropriate for financial signals with moderate signal components spread over diverse frequencies. The rectangular window is the most useful for security return data with high noise levels and therefore flat spectrums. As shown in Table I financial signals have 4-year, 2-year and 1-year 'tones' or frequencies relatively close together with 6-month and quarterly frequencies a long distance apart. The rectangular window gives equal importance to all frequencies and maximizes the resolution.

12.2. TRENDS IN RETURN SIGNALS

An important question in signal analysis is the desirability and feasibility of removing or filtering out trends. In order to improve the quality of the discrete periodic variance estimates and to insure the return process is covariance stationary, some researchers have examined the importance of removing the trend (see Watson, 1986). Large means and linear and higher order trends may bias the low frequency estimates of the variance of the process. However, trend removal can introduce erroneous peaks in the variance spectrum at low frequencies. Removal of the sample mean is particularly undesirable when as in the case of financial return processes, the noise level is high. Trends in mean may be related to the presence of long memory that may in turn be responsible for changing variance (see Granger, 1988).

Mean-variance optimization is very sensitive to errors in the estimates of the inputs. Britten-Jones (1999) found that the magnitude of the error in estimates of the weights of the sample efficient portfolios is large. Chopra and Ziemba (1993) showed that errors in means are eleven times as important as errors in variances and errors in variances are about twice as important as errors in covariance on the composition of the optimal mean-variance portfolio. In turn the value of the mean for a series can be greatly affected by the presence of a single outlier. Trimming or clipping may be use to remove outliers to produce more robust sample estimates (see Kleiner, Martin and Thomson, 1979). No trend removal or clipping was used and all estimates were taken directly from the raw security return time series to develop the inputs to the Digital Portfolio Theory model for the test universe of 20 securities.

13. Comparing Digital Portfolio Theory to Modern Portfolio Theory

This section solves the Digital Portfolio Theory model with no short selling for a small universe of 20 securities and compares the solutions to the efficient mean-variance portfolios. A close approximation to the mean-variance efficient set of portfolios can be found by forcing all periodic risk compenent to be equal. By solving the Digital Portfolio Theory model (12) with the all the right-hand-sides, b_k, set equal, each of the periodic contributions to the risk of the portfolio is constrained at the same level. This approximates the mean-variance efficient frontier since the expression for the portfolio variance (10) will be minimized when all K of the systematic and unsystematic variance components are equal.

$$\operatorname{var}(\tilde{r}_p[n]) = \frac{1}{2}\sum_{k=1}^{K}\left(\left(\sum_{j=1}^{N} W_j R_{kj}\cos\theta_{kmj}\right)^2 + \left(\sum_{j=1}^{N} W_j R_{kj}\sin\theta_{kmj}\right)^2\right) \tag{10}$$

$$\text{Portfolio risk } = \frac{1}{2}\sum_{k=1}^{K}((\text{systematic risk}_{pk})^2 + (\text{unsystematic risk}_{pk})^2)\,.$$

It is only an approximation to Modern Portfolio Theory since different securities display differing levels of the periodic risk components and since the contribution of each risk component to expected return may differ considerably for each security. By constraining each risk component equally we are forcing the solution portfolio's signal to have low autocorrelation and little or no memory.

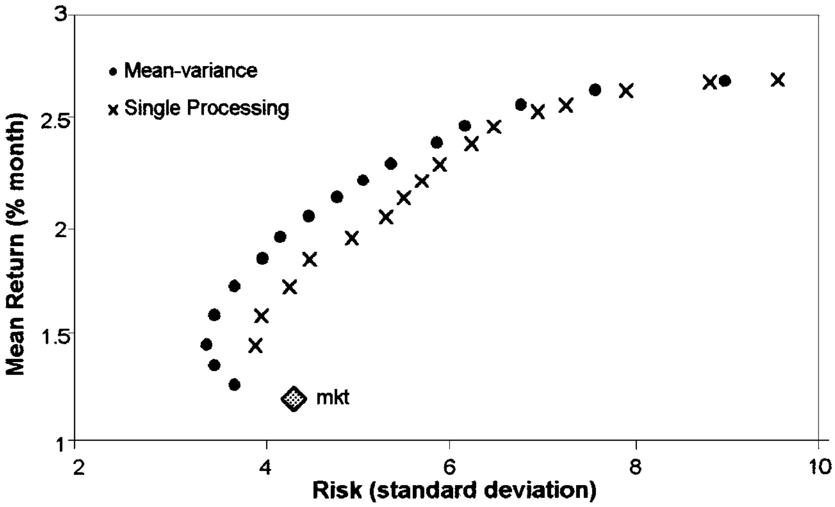

Figure 2. Comparison of efficient frontiers: Modern and Digital Portfolio Theory. The mean-variance efficient frontier compared to the efficient frontier generated by the Digital Portfolio Theory model found by constraining all the periodic portfolio variance components by the same right-hand-side constant.

The no-memory Digital Portfolio Theory efficient portfolios are compared to the mean-variance efficient frontier in Figure 2. In the low risk region, equally constrained Digital Portfolio Theory gives an approximation to the mean-variance efficient set but falls below it in every case. For medium risk the no-memory efficient portfolios found using Digital Portfolio Theory model approach the mean-variance frontier. For the high-risk portfolios, no-memory Digital Portfolio Theory and Modern Portfolio Theory models pick the same securities. For the high-risk investor memory or autocorrelation is not important and the solution is dominated by high return securities.

Figure 3 gives some explanation of the performance. The number of securities selected in the portfolios of the signal-processing model is less than or equal to those selected by the mean-variance model. The smaller efficient portfolios found in equally constrained Digital Portfolio Theory result from constraining risk simultaneously in multiple dimensions rather than constraining risk in only one dimension in Modern Portfolio Theory.

Figure 4 demonstrates one distinct advantage of Digital Portfolio Theory. Efficient zero calendar risk portfolios can be solved for by setting the right-hand side constants for one value of k equal to zero, $b_{sk} = b_{uk} = 0$, while leaving the other right-hand sides unconstrained in Equations (12)–(18). Independently controlling the calendar and non-calendar risk of the solution portfolios is important since traders can select efficient portfolios with risk characteristics that match their expectations, holding periods and trade dates. In Figure 4, the 1-year zero risk portfolio provides lower return than other zero risk calendar portfolios. An

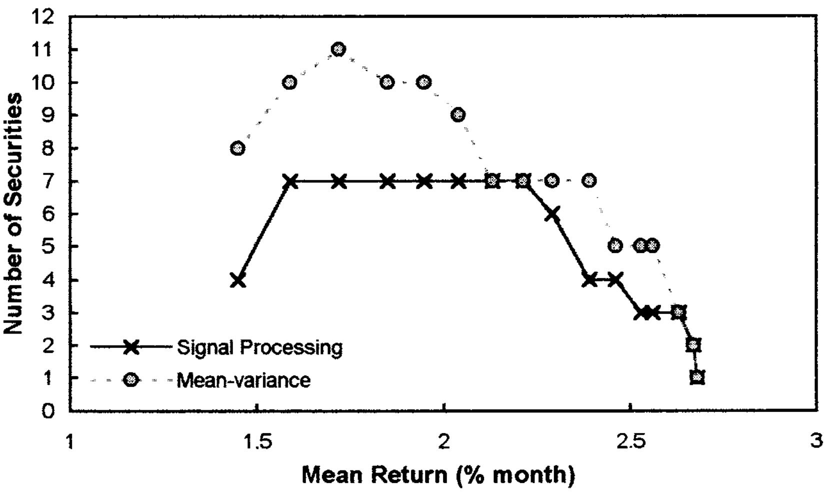

Figure 3. Number of securities in the efficient portfolios: Mean-variance and signal processing. The number of securities in the efficient portfolios of the Digital Portfolio Theory model is less than or equal to the number of securities in the Modern Portfolio Theory efficient portfolios.

investor will have to sacrifice more return to hold the zero yearly risk portfolio than to hold the zero quarterly risk portfolio using this security universe. The zero yearly risk portfolio has less return because yearly risk contributes more to the total risk of these securities than do the other calendar risk components. There was no feasible solution for the zero two-year risk efficient portfolio from this small security universe.

14. Conclusion

The signal processing portfolio selection model is a new approach that offers considerable theoretical and practical advantages. Digital Portfolio Theory offers a portfolio decision model that permits the investor to utilize memory by controlling calendar and non-calendar components of the efficient portfolio's variance. The signal processing description of risk gives a more detailed description of risk that includes autocorrelation contributions. The introduction of the risk phasor or vestor adds a new interpretation to systematic and unsystematic risk and allows complex mathematics to be applied to risk analysis. The traditional mean-variance portfolio selection model assumes that investors have no memory. By breaking the portfolio variance into periodic variance components Digital Portfolio Theory not only allows autocorrelation to be controlled in the efficient portfolio solution but also allows control over systematic and unsystematic components of portfolio risk. In addition, Digital Portfolio Theory reduces the number of terms necessary to represent the covariance information and simplifies the quadratic expression of portfolio variance in Modern Portfolio Theory by replacing it with a set of linear

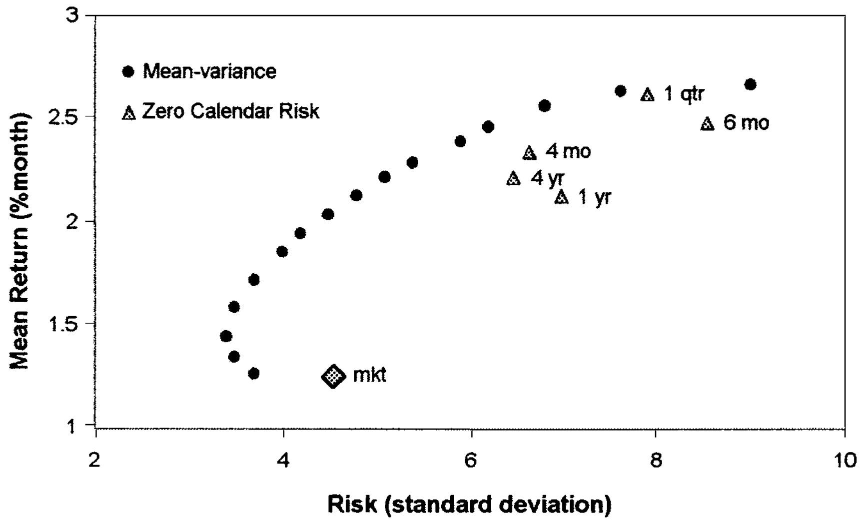

Figure 4. Efficient portfolios with zero calendar risk. The efficient portfolio characteristics using Digital Portfolio Theory to force the periodic variance components of the efficient portfolio to be zero for specific calendar periods.

constraints. The signal-processing model can be applied to large universes and does not require any special structure of the covariance or autocovariance. The maximization LP framework facilitates control over additional fundamental constraints and allows the addition of negative variables and integer variables to model short selling and transaction costs. A comparison of efficient portfolios shows that the signal-processing model can approximate the mean-variance efficient set with smaller portfolios. In addition, it can identify portfolios with zero periodic risk for calendar periods selected, based on the calendar expectations and holding period of the investor. The paper has suggested potential areas for further research such as solving for efficient pure calendar risk portfolios to be used in arbitrage pricing and efficient zero unsystematic risk portfolios for capital market line estimation. Using the signal processing representation of risk it may be possible to apply digital mathematics to more complex financial systems encountered in multiple period contexts.

Acknowledgements

The author is grateful to George B. Dantzig, Fred W. Glover, Clive W.J. Granger, Harry M. Markowitz, Merton H. Miller and Charles S. Tapiero for providing their input. Any errors are the responsibility of the author. Part of this research was supported by a grant from AMOCO.

Notes

[1] The Power Spectral Density (PSD) function of MATLAB can compute Fourier Transforms from return series with calendar signal lengths.

[2] MATLAB uses the Welch method to find the spectral density and allows rectangular and Hanning windows.

[3] Jones (1997) has developed a software package available to researchers based on Digital Portfolio Theory (http://www.portfolionetworks.com). The software package can find efficient portfolios from a universe of 8000 securities and uses the Welch method to measure risk.

References

Alexander, G.J. (1976). The derivation of efficient sets. *Journal of Financial and Quantitative Analysis*, **11**, 817–830.

Andersen, T.G. and Bollerslev, T. (1997). Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns. *Journal of Finance*, **52**(3), 975–1005.

Baillie, R.T. (1996). Long memory processes and fractional integration in economics. *Journal of Economics*, **73**, 5–59.

Beller, K. and Nofsinger, J.R. (1998). On stock return seasonality and conditional heteroskedasticity. *Journal of Financial Research*, **21**(2), 229–246.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**, 637–654.

Blackman R.B. and Tukey, J.W. (1959). *The Measurement of Power Spectra*. Dover Publications, Inc.

Bollerslev, T. and Engle, R.F. (1993). Common persistence in conditional variances. *Econometrica*, **61**, 167–186.

Bollerslev, T. and Mikkelsen, H.O. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, **73**, 151–184.

Booth, J.R. and Booth, L.C. (1999). The presidential cycle anomaly in stock market returns. Working Paper, Arizona State University, Tempe.

Breidt, F.J., Crato, N. and de Lima, P. (1998). The detection and estimation of long memory in stochastic volatility. *Journal of Econometrics*, **83**, 325–348.

Britten-Jones, M. (1999). The sampling error in estimates of mean-variance efficient portfolio weights. *Journal of Finance*, **54**(2), 655–671.

Chen, A.H.Y., Jen, F.C. and Zionts, S. (1971). The optimal portfolio revision policy. *The Journal of Business*, 51–61.

Chopra, V.K. and Ziemba, W.T. (1993). The effect of errors in means, variance, and covariances on optimal portfolio choice. *Journal of Portfolio Management*, Winter, 6–11.

Cohen, K.J. and Pogue, G.A. (1967). An empirical evaluation of alternative portfolio selection models. *Journal of Business*, **40**, 166–193.

Cooper, R. (1974). Efficient capital markets and the quantity theory of money. *Journal of Finance*, **29**, 887–908.

Crum, R.L., Klingman, D.D. and Travis, T.A. (1979). Implementation of large-scale financial planning models: Solution efficient transformations. *Journal of Financial and Quantitative Analysis*, **14**, 137–152.

Dantzig, G.B. and Infanger, G. (1993). Multi-stage stochastic linear programs for portfolio optimization. *Annuals of Operations Research*, **45**, 59–76.

De Santis, G. and Gerard, B. (1997). International asset pricing and portfolio diversification with time-varying risk. *Journal of Finance*, **52**(5), 1881–1912.

Dumas, B. and Luciano, E. (1991). An exact solution to a dynamic portfolio choice problem under transaction costs. *Journal of Finance*, **46**, 577–595.

Durlauf, S.W. (1991). Spectral based testing of the martingale hypothesis. *Journal of Econometrics*, **50**, 355–376.

Elton, E.J. and Gruber, M.J. (1973). Estimating the dependence structure of share prices – implications for portfolio selection. *Journal of Finance*, **28**, 1203–1233.

Elton, E.J. and Gruber, M.J. (1974a). The multi-period consumption investment problem and single period analysis. *Oxford Economics Papers*, Sept., 289–595.

Elton, E.J. and Gruber, M.J. (1974b). On the optimality of some multiperiod portfolio selection criteria. *Journal of Business*, **47**, 231–243.

Elton, E.J. and Gruber, M.J. (1975). *Finance as a Dynamic Process*. Prentice Hall, Englewood Cliffs, NJ.

Elton, E.J., Gruber, M.J. and Padberg, M.W. (1976). Simple criteria for optimal portfolio selection. *Journal of Finance*, **31**(5), 1341–1357.

Elton, E.J., Gruber, M.J. and Padberg, M.W. (1977). Simple criteria for optimal portfolio selection with upper bounds. *Operations Research*, **25**, 952–967.

Fama, E.F. (1970). Multiperiod consumption-investment decisions. *American Economic Review*, **60**, 163–174.

Fama, E.F. (1991). Efficient capital markets: II. *Journal of Finance*, **46**(5), 1575–1617.

Fama, E.F. and French, K.R. (1988). Permanent and temporary components of stock prices. *Journal of Political Economy*, **96**(2), 246–273.

Glover, F. and Jones, C.K. (1988). A stochastic generalized network model and large-scale algorithm for portfolio selection. *Journal of Information and Optimization Science*, **9**, 299–316.

Granger, C.W.J. (1988). Models that generate trends. *Journal of Time Series Analysis*, **9**(4), 329–343.

Granger, C.W.J. (1992). Forecasting stock market prices: Lessons for forecasters. *International Journal of Forecasting*, **8**, 3–13.

Granger, C.W.J. and Ding, Z. (1996). Varieties of long memory models. *Journal of Econometrics*, **73**, 61–77.

Granger, C.W.J. and Morgenstern, O. (1970). *Predictability of Stock Market Prices.* D.C. Heath Company, Mass.

Grauer, R.R. and Hakansson, N.H. (1993). On the use of mean-variance and quadratic approximations in implementing dynamic investment strategies: A comparision of returns and investment policies. *Management Science*, **39**, 856–871.

Grinblatt, M. and Titman, S. (1987). The relation between mean-variance efficiency and arbitrage pricing. *Journal of Business*, **60**(1), 97–112.

Grinold, R.C. (1999). Mean-variance and scenario-based approaches to portfolio selection. *Journal of Portfolio Management*, Winter, 10–22.

Gunthorpe, D. and Levy, H. (1994). Portfolio composition and the investment horizon. *Financial Analysts Journal*, January–February, 51–56.

Hansen, L.P. and Richard, S.F. (1987). The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. *Econometrica*, **55**(3), 587–613.

Harkansson, N. (1971a). Multi-period mean-variance analysis: Toward a general theory of portfolio choice. *Journal of Finance*, **26**, 857–884.

Harkansson, N. (1971b). On optimal myopic portfolio policies, with and without serial correlation of yields. *Journal of Business*, **44**, 324–334.

Harkansson, N. (1979). A characterization of optimal multi-period portfolio policies. *TIMS Studies in the Management Sciences*, **11**, 169–177.

Harris, F.J. (1978). On the use of windows for harmonic analysis with the discrete Fourier transform. *Proceedings of the IEEE*, **66**(1), 51–83.

Harvey, C.R. (1989). Time-varying conditional covariances in tests of asset pricing models. *Journal of Financial Economics*, **24**, 289–317.

Hensel, C.R. and Ziemba, W.T. (1996). Investment results from exploiting turn-of-the-month effects. *Journal of Portfolio Management*, Spring, 17–23.

Jenkins, G.M. and Watts, D.G. (1969). *Spectral Analysis and Its Applications*. Holden-Day.

Jones, C.K. (1992). *Portfolio Management: New Models for Successful Investment Decisions*. McGraw Hill, Inc., London.

Jones, C.K. (1997). *PSS Release 2.0: Digital Portfolio Theory*. Portfolio Selection Systems, Gainesville, Florida, http://www.portfolionetworks.com.

Keim, D.B. (1983). Size-related anomalies and stock return seasonality. *Journal of Financial Economics*, **12**, 13–32.

Kleiner, B., Martin, R.D. and Thomson, D.J. (1979). Robust estimation of power spectra. *Journal of the Royal Statistical Society B*, **41**(3), 313–351.

Kroll, Y., Levy, H. and Markowitz, H.M. (1984). Mean-variance versus direct utility maximization. *Journal of Finance*, **39**(1), 47–61.

Lemke, C.E. (1965). Bimatrix equilibrium points and mathematical programming. *Management Science*, **11**, 681–689.

Li, D. and Ng, W.L. (2000). Optimal dynamic portfolio selection: Multiperiod mean-variance formulation. *Mathematical Finance*, **10**(3), July, 387–406.

Linn, S.C. and Lockwood, L.J. (1988). Short-term stock price patterns: NYSE, AMEX, OTC. *Journal of Portfolio Management*, Winter, 30–34.

Markowitz, H.M. (1952). Portfolio selection. *Journal of Finance*, **7**, 77–91.

Markowitz, H.M. (1956). The optimization of a quadratic function subject to linear constraints. *Naval Research Logistics Quarterly*, **3**, 111–133.

Markowitz, H.M. and Perold, A.F. (1981). Portfolio analysis with factors and scenarios. *Journal of Finance*, **36**(14), 871–877.

Markowitz, H.M., Todd, P., Xu, G. and Yamane, Y. (1992). Fast computation of mean-variance efficient sets using historical covariances. *Journal of Financial Engineering*, **1**(2), 117–132.

Merton, R.C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, **51**, 247–257.

Merton, R.C. (1973). The theory of rational option pricing. *Bell Journal of Economics and Management Science*, **4**, 141–183.

Marple, S.L. (1987). *Digital Spectral Analysis*. Prentice-Hall, Inc.

Modigliani, F. and Miller, M. (1958). The cost of capital, corporation finance, and the theory of investment. *American Economic Review*, **48**, 261–297.

Mossin, J. (1968). Optimal multiperiod portfolio policies. *Journal of Business*, 215–229.

Mulvey, J.M. (1987). Nonlinear network models in finance. In *Advances in Mathematical Programming and Financial Planning*, **1**, 253–262. JAI Press.

Mulvey, J.M. and Vladimirou, H. (1989). Stochastic network optimization models for investment planning. *Annuals of Operations Research*, **20**, 187–217.

Mulvey, J.M. and Vladimirou, H. (1991). Applying the progressive hedging algorithm to stochastic generalized networks. *Annuals of Operations Research*, **31**, 399–424.

Mulvey, J.M. and Vladimirou, H. (1992). Stochastic network programming for financial planning problems. *Management Science*, **38**(11), 1642–1664.

Mulvey, J.M. and Ziemba, W.T. (1998). Asset and liability management systems for long-term investors: Discussion of the issues. *Worldwide Asset and Liability Modeling*, 3–38. Cambridge University Press, Cambridge, England.

Nau, R.F. and McCardle, K.F. (1991). Arbitrage, rationality, and equilibrium. *Theory and Decision*, **31**, 199–240.

Ostermark, R. (1991). Vector forecasting and dynamic portfolio selection: Empirical efficiency of recursive multiperiod strategies. *European Journal of Operational Research*, **55**, 46–56.

Pang, J.S. (1980). A parametric linear complementary technique for optimal portfolio selection with a risk-free asset. *Operations Research*, **28**, 927–941.

Parzen, E. (1957). On choosing an estimate of the spectral density function of a stationary time series. *Annals of Mathematical Statistics*, **28**, 921–932.

Penman, S.H. (1987). The distribution of earnings news over time and seasonalities in aggregate stock returns. *Journal of Financial Economics*, **18**, 199–228.

Perold, A.F. (1984). Large-scale portfolio optimization. *Management Science*, **30**(10), 1143–1160.

Poterba, J.M. and Summers, L.H. (1988). Mean reversion in stock prices: Evidence and implications. *Journal of Financial Economics*, **22**, 27–59.

Praetz, P. (1979). Testing for a plat spectrum on efficient market price data. *Journal of Finance*, **34**, 645–658.

Rosenberg, B. (1974). Extra-market components of covariance in security returns. *Journal of Financial and Quantitative Analysis*, **9**, 263–274.

Ross, S. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, **13**, 341–360.

Rozeff, M.S. and Kinnery, W.R. (1976). Capital market seasonality: The case of stock returns. *Journal of Financial Economics*, **3**, 379–402.

Samuelson, P.A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics*, **51**, 239–246.

Sharpe, W.F. (1971). A linear programming application for the general portfolio analysis problem. *Journal of Financial and Quantitative Analysis*, **6**, 1263–1275.

Sharpe, W.F. (1963). A simplified model for portfolio analysis. *Management Science*, **9**, 277–293.

Smith, K.V. (1967). A transition model for portfolio revision. *Journal of Finance*, **22**, 425–439.

Thorley, S.R. (1995). The time-diversification controversy. *Financial Analysts Journal*, May–June, 68–76.

Wachtel, S.B. (1942). Certain observations on seasonal movements in stock prices. *Journal of Business*, **15**(2), 184–193.

Watson, M.W. (1986). Univariate detrending methods with stochastic trends. *Journal of Monetary Economics*, **18**, 49–75.

Welch, P.D. (1967). The use of fast Fourier transforms for the estimation of power spectra: A method based on time averaging over short modified periodograms. *IEEE Transactions in Audio Electroacoustics*, **AU-15**, June, 70–73.

Wiener, N. (1930). Generalized harmonic analysis. *Acta Math.*, **55**, 117–258.

Wiener, N. (1970). *Time Series*. M.I.T. Press.

Winkler, R.L. and Barry, C.B. (1975). A Bayesian model for portfolio selection and revision. *Journal of Finance*, **30**, 179–192.

Wolfe, P. (1959). The Simplex method for quadratic programming. *Econometrica*, **27**, 382–398.